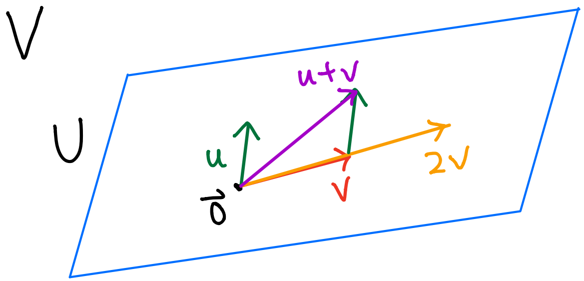
Math 2040 C Week 2

Let V be a vector space over $\mathbb{F}$.

__Defn__ A subset $U \subseteq V$ is called a subspace of V if U is also a vector space (under the same $+, \cdot$ in V)



__Prop__ Let $U \subseteq V$ be a subset. Then

U is a subspace of $V \Leftrightarrow$ (i)-(iii) hold :

(i) $\vec{0} \in U$

(ii) $u, v \in U \Rightarrow u+v \in U$ (closed under addition)

(iii) $v \in U, \lambda \in \mathbb{F} \Rightarrow \lambda v \in U$ (closed under scalar multiplication)

__Pf__ ($\Rightarrow$) clear from defn of vector space.

($\Leftarrow$) Suppose (i)-(iii) hold for U. Then

$$\left\{ \begin{array}{l} + : U \times U \longrightarrow U \\ \cdot : \mathbb{F} \times U \longrightarrow U \end{array} \right. \quad \text{are well-defined on } U$$

(VS1), (VS2), (VS5)-(VS7) hold for V

$\Rightarrow$ they hold for U

(i) $\Rightarrow$ (VS3) hold for V

Also, for any $v \in U$, (iii) $\Rightarrow -v = (-1)v \in U$.

$\therefore$ (VS4) hold for V

Hence U is a vector space

__Rmk__ The smallest and largest subspace of V is __zero subspace__ $\{0\}$ and V respectively.

In other words, for any subspace $U \subseteq V$,

$$\{0\} \subseteq U \subseteq V$$

Example of subspaces

① $\{(x_1,x_2,x_3)\in\mathbb{F}^3 : x_1-x_2+2x_3=b\}$ is a subspace of $\mathbb{F}^4 \Leftrightarrow b=0$

② $P_m(\mathbb{F})$ is a subspace of $\mathbb{P}(\mathbb{F})$

③ $\{f:(0,1)\to\mathbb{R} : f \text{ is differentiable}\}$
$\subseteq \{f:(0,1)\to\mathbb{R} : f \text{ is continuous}\}$
$\subseteq \mathbb{R}^{(0,1)}$ are subspaces

④ Let $n\in\mathbb{N}$. Then the subset of symmetric matrices $\{A\in M_{n\times n}(\mathbb{R}) : A=A^t\}$ is a subspace of $M_{n\times n}(\mathbb{R})$.

⑤ $\{(x_1,x_2,\cdots)\in\mathbb{R}^\infty : \lim_{n\to\infty} x_n = b\}$ is a subspace of $\mathbb{R}^\infty \Leftrightarrow b=0$

Pf of ⑤

Let $\mathbb{R}_b^\infty=\{(x_1,x_2,\cdots)\in\mathbb{R}^\infty : \lim_{n\to\infty} x_n=b\}$

($\Rightarrow$) Suppose $\mathbb{R}_b^\infty$ is a subspace. The $\vec{0}=(0,0,0,\cdots)\in\mathbb{R}_b^\infty$

Hence, $b=\lim_{n\to\infty}\vec{0}_n=\lim_{n\to\infty}0=0$

($\Leftarrow$) We need show $\mathbb{R}_0^\infty$ contains $\vec{0}$ and is closed under $+,\cdot$.

(i) $\lim_{n\to\infty}\vec{0}_n=0 \Rightarrow \vec{0}\in\mathbb{R}_0^\infty$

(ii) Suppose $x=(x_1,x_2,\cdots)$, $y=(y_1,y_2,\cdots)\in\mathbb{R}_0^\infty$. Then

$$\lim_{n\to\infty}(x+y)_n=\lim_{n\to\infty}(x_n+y_n)=\lim_{n\to\infty}x_n+\lim_{n\to\infty}y_n$$
$$=0+0=0$$

$\Rightarrow x+y\in\mathbb{R}_0^\infty$

(iii) Suppose $x=(x_1,x_2,\cdots)\in\mathbb{R}_0^\infty$, $\lambda\in\mathbb{R}$. Then

$$\lim_{n\to\infty}(\lambda x)_n=\lim_{n\to\infty}\lambda x_n=\lambda\lim_{n\to\infty}x_n=\lambda(0)=0$$

$\Rightarrow \lambda x\in\mathbb{R}_0^\infty$

Hence, $\mathbb{R}_0^\infty$ is a subspace of $\mathbb{R}^\infty$

eg Which of the following subsets are subspaces?

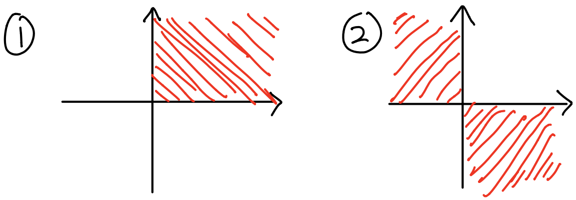
① $\{(x_1,x_2)\in\mathbb{R}^2 : x_1,x_2 \geq 0\} \subseteq \mathbb{R}^2$

② $\{(x_1,x_2)\in\mathbb{R}^2 : x_1 x_2 \leq 0\} \subseteq \mathbb{R}^2$

③ $\{f:\mathbb{R}\to\mathbb{R} : f''(t) = f'(t) - 3f(t)\} \subseteq \mathbb{R}^{\mathbb{R}}$

④ The subset of skew-symmetric matrices

$\{A\in M_{n\times n}(\mathbb{R}) : A = -A^t\} \subseteq M_{n\times n}(\mathbb{R})$



	contains $\vec{0}$?	closed under + ?	closed under · ?
①	✓	✓	✗
②	✓	✗	✓
③	✓	✓	✓
④	✓	✓	✓

Ex Prove rigorously

eg Suppose $U_1, U_2 \subseteq V$ are subspaces, show that

① $U_1 \cap U_2$ is also a subspace of V.

② $U_1 \cup U_2$ may not be subspace by a counter-example

Sol

① U_1, U_2 are subspaces $\Rightarrow \vec{0}\in U_1, U_2 \Rightarrow \vec{0}\in U_1\cap U_2$

Also, for any $u_1,u_2 \in U_1\cap U_2$, we have

$u_1,u_2 \in U_1, U_2 \Rightarrow u_1+u_2 \in U_1, U_2$ ($\because U_1, U_2$ are subspaces)

$\Rightarrow u_1+u_2 \in U_1\cap U_2$

$\therefore$ $U_1\cap U_2$ is closed under addition.

Finally, for any $u\in U_1\cap U_2$ and $\lambda\in\mathbb{F}$, we have

$u\in U_1,U_2 \Rightarrow \lambda u \in U_1, U_2$ ($\because U_1, U_2$ are subspaces)

$\Rightarrow \lambda u \in U_1\cap U_2$

$\therefore$ $U_1\cap U_2$ is closed under scalar multiplication.

Hence $U_1\cap U_2$ is a subspace.

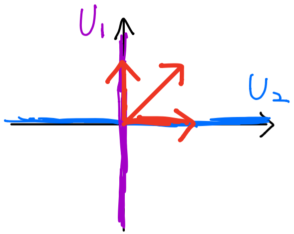
② Consider $U_1 = \{(x,0) : x \in \mathbb{R}\}$, $U_2 = \{(0,y) : y \in \mathbb{R}\}$

Then $(1,0) \in U_1 \subset U_1 \cup U_2$ $\quad (0,1) \in U_2 \subset U_1 \cup U_2$

but $(1,0) + (0,1) = (1,1) \notin U_1 \cap U_2$

$\therefore$ $U_1 \cup U_2$ is not closed under addition

$\Rightarrow$ $U_1 \cup U_2$ is not a subspace.



Rmk

Similarly, the intersection of any collection of subspaces is also a subspace

Sum of subspaces

Defn Let $U_1, U_2, \cdots, U_m$ be subsets of V. Then their sum is defined to be

$$U_1 + U_2 + \cdots + U_m = \{u_1 + u_2 + \cdots + u_m : u_i \in U_i \text{ for } i = 1, 2, \cdots, m\}$$

Prop Suppose $U_1, U_2, \cdots, U_m$ are subspace of V. Then $U_1 + U_2 + \cdots + U_m$ is the smallest subspace of V containing $U_1, U_2, \cdots, U_m$

Pf

Easy to show $U_1 + U_2 + \cdots + U_m$ is a subspace of V (Ex)

Also, for any $u_1 \in U_1$, since $\vec{0} \in U_2, U_3, \cdots, U_m$, we have

$$u_1 = u_1 + \vec{0} + \vec{0} + \cdots + \vec{0} \in U_1 + U_2 + \cdots + U_m$$

$\Rightarrow$ $U_1 \subseteq U_1 + U_2 + \cdots + U_m$.

Similarly, $U_i \subseteq U_1 + U_2 + \cdots + U_m$ for any $i = 1, 2, \cdots, m$

Finally, suppose W is a subspace containing $U_1, U_2, \cdots, U_m$

Then for any $u_1 + u_2 + \cdots + u_m \in U_1 + U_2 + \cdots + U_m$

$u_i \in U_i \subseteq W \ \forall i \Rightarrow \sum_{i=1}^{m} u_i \in W$ since W is a subspace

$\therefore$ $\quad U_1 + U_2 + \cdots + U_m \subseteq W$

Defn Let $U_1, U_2, \cdots, U_m$ be subspace of V. Then their sum $U_1 + U_2 + \cdots + U_m$ is called a <u>direct sum</u> if any element u in it can be expressed as

$$u = u_1 + u_2 + \cdots + u_m \quad \text{with } u_i \in U_i$$

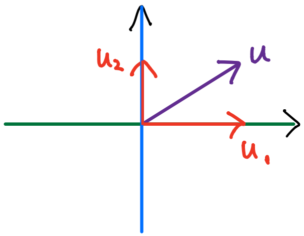
in a <u>unique</u> way.

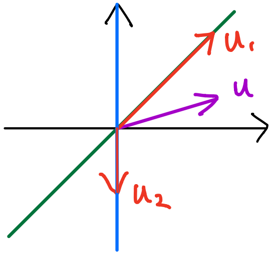
If $U_1 + U_2 + \cdots + U_m$ is a direct sum, we write it as $U_1 \oplus U_2 \oplus \cdots \oplus U_m$

eg

$$\mathbb{R}^2 = \{(x,0) : x \in \mathbb{R}\} \oplus \{(0,y) : y \in \mathbb{R}\}$$

$$= \{(s,s) : s \in \mathbb{R}\} \oplus \{(0,y) : t \in \mathbb{R}\}$$





u_1, u_2 are unique (no other choices)

eg Let $U_1 = \{(x,y,0) : x,y \in \mathbb{R}\}$, $U_2 = \{(0,y,z) : y,z \in \mathbb{R}\}$

Show that $\mathbb{R}^3 = U_1 + U_2$ but is not a direct sum

Sol Clearly, $U_1, U_2 \subseteq \mathbb{R}^3 \Rightarrow U_1 + U_2 \subseteq \mathbb{R}^3$

Also, for any $(x,y,z) \in \mathbb{R}^3$,

$$(x,y,z) = (x,y,0) + (0,0,z)$$

where $(x,y,0) \in U_1$ and $(0,0,z) \in U_2$

$\Rightarrow \mathbb{R}^3 \subseteq U_1 + U_2$

$\therefore \mathbb{R}^3 = U_1 + U_2$

To show it is not a direct sum, note that

$$(1,2,3) = (1,2,0) + (0,0,3)$$
$$= (1,0,0) + (0,2,3)$$

decomposition is not unique

where $(1,2,0), (1,0,0) \in U_1$

$(1,0,0), (0,2,3) \in U_2$

$\therefore U_1 + U_2$ is not a direct sum

Prop 1.44

Suppose $U_1, U_2, \cdots U_m$ are subspaces of V. Then

① $U_1 + U_2 + \cdots + U_m$ is a direct sum $\Leftrightarrow$

② $\vec{0} = u_1 + u_2 + \cdots + u_m$ with $u_i \in U_i \ \forall i$

only if $u_1 = u_2 = \cdots = u_m = \vec{0}$

Pf ($\Rightarrow$) Assume ①. Note that

$\vec{0} = u_1 + u_2 + \cdots + u_m$ with $u_i = \vec{0} \in U_i \ \forall i$

Defn of direct sum and ①

$\Rightarrow$ this decomposition is unique $\Rightarrow$ ②

($\Leftarrow$) Assume ②

let $v \in U_1 + U_2 + \cdots + U_m$ with

$$v = u_1 + u_2 + \cdots + u_m \quad (*)$$
$$= w_1 + w_2 + \cdots + w_m,$$

where $u_i, w_i \in U_i \ \forall i$.

We want to show $u_i = w_i \ \forall i$

Note that

$\vec{0} = v - v = (u_1 - w_1) + (u_2 - w_2) + \cdots + (u_m - w_m)$

and $u_i - w_i \in U_i$

So ② $\Rightarrow u_i - w_i = \vec{0} \ \forall i \Rightarrow u_i = w_i \ \forall i$

$\therefore$ the decomposition $(*)$ is unique

$\therefore U_1 + U_2 + \cdots + U_m$ is a direct sum

Prop 1.45 Suppose U, W are subspaces of V.

Then $U + W$ is a direct sum $\Leftrightarrow U \cap W = \{0\}$

Pf ($\Rightarrow$) Suppose $U+W$ is a direct sum.

Let $v \in U \cap W$. Then $v = v + 0$ with $v \in U$, $0 \in W$

$= 0 + v$ with $0 \in U$, $v \in W$

$U + W$ is a direct sum $\Rightarrow$ unique decomposition

$\Rightarrow v = 0$

$\Rightarrow U \cap W \subseteq \{0\}$

Clearly, $\{0\} \subseteq U \cap W \Rightarrow U \cap W = \{0\}$

$(\Leftarrow)$ Suppose $U \cap W = \{0\}$

Let $\vec{0} = u + w$, where $u \in U$, $w \in W$

Then $w = -u = (-1)u \in U$

$\Rightarrow w \in U \cap W = \{0\} \Rightarrow w = 0$

$\therefore u = -w = 0$

Prop 1.44 $\Rightarrow$ $U + W$ is a direct sum

> <u>Rmk</u> For $m > 2$
>
> $\bigcap_{i=1}^{m} U_i = \{0\}$ $\cancel{\Rightarrow}$ $U_1 + U_2 + \cdots + U_m$
>
> $\Leftarrow$ is a direct sum

<u>eg</u> Let $V = \mathbb{F}^2$ and

$U_1 = \{(x, 0) : x \in \mathbb{F}\}$

$U_2 = \{(0, y) : y \in \mathbb{F}\}$

$U_3 = \{(s, s) : s \in \mathbb{F}\}$

Then $U_1 \cap U_2 \cap U_3 = \{0\}$

and $U_1 + U_2 + U_3$ is not a direct sum

<u>Examples of direct sums</u>

① Let $n \in \mathbb{N}$ and $V = M_{n \times n}(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

$$V = \{A \in V : \underbrace{A^t = A}_{\text{symmetric matrix}}\} \oplus \{A \in V : \underbrace{A^t = -A}_{\text{Skew-symmetric matrix}}\}$$

② Let $\mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$

U_e = the subspace of even functions

$= \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = f(x) \ \forall x \in \mathbb{R}\}$

U_o = the subspace of odd functions

$= \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x) \ \forall x \in \mathbb{R}\}$

Then $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$

③ Let $U_1 = \{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 0\}$

$U_2 = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a constant function}\}$

Then $\mathbb{R}^{\mathbb{R}} = U_1 \oplus U_2$